W51. In all triangle ABC holds:

1). $\sum\left(\frac{2a^3}{b^3}+\frac{b^3}{c^3}\right)r_a^2 \geq \frac{12s^2(2R-r)^2}{s^2+4Rr+r^2}$

2). $\sum\left(\frac{2r_a^3}{r_b^3}+\frac{r_b^3}{r_c^3}\right)a^2 \geq 12(2R-r)^2$

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1. Since by AM-GM Inequality $\frac{2a^3}{b^3}+\frac{b^3}{c^3} \geq 3\sqrt[3]{\left(\frac{a^3}{b^3}\right)^2 \cdot \frac{b^3}{c^3}} = \frac{3a^2}{bc}$ and $r_a = \frac{sr}{s-a}$

then $\sum\left(\frac{2a^3}{b^3}+\frac{b^3}{c^3}\right)r_a^2 \geq \sum \frac{3a^2}{bc} \cdot \frac{s^2r^2}{(s-a)^2} = 3s^2r^2\sum \frac{a^2}{(s-a)^2} \cdot \frac{1}{bc}$.

Also, by Cauchy Inequality we have $\sum bc \cdot \sum \frac{a^2}{(s-a)^2} \cdot \frac{1}{bc} \geq \left(\sum \frac{a}{s-a}\right)^2$.

Noting that $\sum \frac{a}{s-a} = \sum \frac{s}{s-a} - 3 = \frac{s\sum(s-b)(s-c)}{(s-a)(s-b)(s-c)} - 3 =$

$\frac{\sum(s^2-s(b+c)+bc)}{r^2} - 3 = \frac{ab+bc+ca-s^2}{r^2} - 3$ and $ab+bc+ca = s^2+4Rr+r^2$

we obtain $\sum \frac{a}{s-a} = \frac{ab+bc+ca-s^2}{r^2} - 3 = \frac{s^2+4Rr+r^2-s^2-3r^2}{r^2} = \frac{2(2R-r)}{r}$

and, therefore, $3s^2r^2\sum \frac{a^2}{(s-a)^2} \cdot \frac{1}{bc} \geq \frac{3s^2r^2\left(\sum \frac{a}{s-a}\right)^2}{ab+bc+ca} =$

$\frac{3s^2r^2}{s^2+4Rr+r^2} \cdot \frac{4(2R-r)^2}{r^2} = \frac{12s^2(2R-r)^2}{s^2+4Rr+r^2}$.

2. And again, since by AM-GM Inequality $\left(\frac{2r_a^3}{r_b^3}+\frac{r_b^3}{r_c^3}\right)a^2 \geq$

$3\sqrt[3]{\left(\frac{r_a^3}{r_b^3}\right)^2 \cdot \frac{r_b^3}{r_c^3}} = \frac{3r_a^2}{r_br_c} = \frac{3\left(\frac{sr}{s-a}\right)^2}{\frac{sr}{s-b} \cdot \frac{sr}{s-c}} = \frac{3(s-b)(s-c)}{(s-a)^2}$ then

$\sum\left(\frac{2r_a^3}{r_b^3}+\frac{r_b^3}{r_c^3}\right)a^2 \geq 3\sum \frac{a^2(s-b)(s-c)}{(s-a)^2}$.

Also, by Cauchy Inequality we have

$\sum \frac{1}{(s-b)(s-c)} \cdot \sum \frac{a^2}{(s-a)^2} \cdot (s-b)(s-c) \geq \left(\sum \frac{a}{s-a}\right)^2 \Leftrightarrow$

$\frac{\sum(s-a)}{(s-a)(s-b)(s-c)} \cdot \sum \frac{a^2(s-b)(s-c)}{(s-a)^2} \geq \left(\sum \frac{a}{s-a}\right)^2 \Leftrightarrow$

$\frac{s}{(s-a)(s-b)(s-c)} \cdot \sum \frac{a^2(s-b)(s-c)}{(s-a)^2} \geq \left(\sum \frac{a}{s-a}\right)^2 \Leftrightarrow$

$\frac{1}{r^2}\sum \frac{a^2(s-b)(s-c)}{(s-a)^2} \geq \left(\frac{2(2R-r)}{r}\right)^2 \Leftrightarrow$

$\sum \frac{a^2(s-b)(s-c)}{(s-a)^2} \geq 4(2R-r)^2$. Hence, $\sum\left(\frac{2r_a^3}{r_b^3}+\frac{r_b^3}{r_c^3}\right)a^2 \geq 12(2R-r)^2$.